(1) State and prove the Bolzona Weierstrass theorem for $\mathbb{R}^n$ (for $n > 1$). (2+4=6 marks)

Statement: If a bounded set S in $\mathbb{R}^n$ contains infinitely many points, then there is at least one point in $\mathbb{R}^n$ which is a limit point of S.
Proof: Since S is bounded, S lies in some ball of radius $r > 0$ and hence lies in

$$J_1 = I_1^{(1)} \times I_2^{(1)} \times \cdots I_n^{(1)} = \{(x_1, x_2, \cdots, x_n) : -r \leq x_j \leq r \ \forall \ 1 \leq j \leq n\}.$$

Each interval $I_j^{(1)}$ can be subdivided into two intervals $I_{j,1}^{(2)} = \{x_j : -r \leq x_k \leq 0\}$ and $I_{j,2}^{(2)} = \{x_j : 0 \leq x_k \leq r\}$. There are 2^n sets $I_{1,k_1}^{(1)} \times \cdots \times I_{n,k_n}^{(1)}$ for some $k_i = 1, 2$. The union of these 2^n sets is J_1. S contains infinitely many points, so does J_1 and hence one of the 2^n sets, say $J_2 = I_1^{(2)} \times \cdots \times I_n^{(2)}$ must contain infinitely many points of S. Similarly bisecting $I_n^{(2)}$ and repeating the process, we obtain J_m as $n-$cartesian product $I_1^{(m)} \times I_n^{(m)}$ of intervals of length $2^{2-m}r$. Let $I_j^{(m)} = [a_j^{(m)}, b_j^{(m)}]$. Since $b_j^{(m)} - a_j^{(m)} = 2^{2-m}r$, we have

$$\sup_m a_j^{(m)} = \inf_m b_j^{(m)} = t_j \ (say).$$

Since $t = (t_1, \cdots, t_n) \in J_i \ \forall i$, any ball $B_\epsilon(t)$ of radius $\epsilon > 0$ centered at $t = (t_1, \cdots, t_n)$ contains J_m for m such that $2^{2-m}r < \epsilon/2$. But by the choice of $J_i's$, there are infinitely many points of S in J_i. Hence $B_\epsilon(t)$ contains infinitely many points of S. Thus t is a limit point of S.
Reference: Apostol's Analysis.

(2) Let $n \geq 1$ be an integer. Consider the three metrics on $\mathbb{R}^n$, the l^1, l^2 and l^∞ metrics. Prove that the topologies on $\mathbb{R}^n$ induced by these three metrics are the same. (10 marks)

Proof: Let $|.|_1, |.|_2, |.|_\infty$ denote the norm in $l_!, l_2, l_\infty$ respectively with respective metrics d_1, d_2, d_∞. Let τ_i denote the topology generated by the finite closures and arbitrary unions over all sets in the basis $\mathcal{B}_i = \{x \in \mathbb{R}^n : |x|_i < r\}, \ \forall \ i = 1, 2, \infty$.

It is enough to prove that the Basis are same in order to prove that the topologies are same: Suppose τ_i, τ_j are topologies with the same basis $\mathcal{B}_i = \mathcal{B}_j$ but $\tau_1 \neq \tau_j$. Suppose $V \in \tau_1$. There exists a collection of open sets $U_\alpha \in \mathcal{B}_i = \mathcal{B}_j$ such that $V = \cup_\alpha U_\alpha \in \tau_j$, since $\mathcal{B}_i = \mathcal{B}_j$.

Now we prove that $\mathcal{B}_i = \mathcal{B}_j$ for all $i, j = 1, 2, \infty$. The norms are defined as

$$\begin{aligned} |x|_i &= (|x_1|^i + \cdots + |x_n|^i)^{1/i} \ \forall \ i = 1, 2 \\ |x|_\infty &= \sup_j \{|x_j|\} \end{aligned}$$

We have

$$|x|_\infty = sup_j\{|x_j|\} \le (\sum_{j=1}^{n} |x_j|^i)^{1/i} \le n^{1/i}|x|_\infty$$

for all $i = 1, 2$. Hence $\mathcal{B}_i \subset \mathcal{B}_\infty \subset \mathcal{B}_i$ for all $i = 1, 2$, that is $\mathcal{B}_1 = \mathcal{B}\infty = \mathcal{B}_2$.

(3) True or False (give reasons):

(a) $\mathbb{Q}$ (with standard metric) is connected. (3 marks)

False. With the standard metric, any open set in $\mathbb{Q}$ is of the form $S = (a, b) \cap \mathbb{Q}$ (or union of such sets). Let c be an irrational number between a and b. Then $S = S_1 \cap S_2$ where $S_1 = (a, c) \cap \mathbb{Q}$ and $S_2 = (c, b) \cap \mathbb{Q}$. Hence $\mathbb{Q}$ is not connected.

(b) Any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous. (3 marks)

True. Suppose $f(x)$ is given by Ax where the matrix satisfies $\|A\| = 0$. Then clearly it is uniformly continuous. Suppose $\|A\| \neq 0$. Then for every $\epsilon > 0$, choose $\delta = \epsilon\|A\|^{-1}$, we get $|f(x) - f(y)| = |Ax - Ay| \le \|A\||x - y| < \epsilon$ whenever $|x - y| < \delta$. Hence f is uniformly continuous.

(c) Any real valued continuous function on a compact metric space has a maximum and a minimum. (3 marks)

True. Let $f : X \to \mathbb{R}$ be a real valued continuous function on a compact metric space X. Continuous image of a compact set is compact. Any compact set in $\mathbb{R}$ is closed and bounded. Hence $f(X)$ is bounded. Also since $f(X)$ is closed both infimum (minimum) and supremum (maximum) are obtained.

(d) A continuous map from a compact metric space to any metric space is uniformly continuous. (3 marks)

True. Let $f : X \to Y$ be a continuous map on a compact metric space X. Since f is continuous, for every $\epsilon > 0$ there exists a δ_x such that $|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \epsilon$. Now $\cup_x B_{\delta_x}(x)$ is an open cover of X. X is compact implies that there are finitely many $\delta_i's$ among all $\delta_x's$ such that $\cup_{i=1}^N B_{\delta_i}(x_i)$ cover X. Choose $\delta = min\{\delta_i\}_i$. Hence the uniform continuity.

(4) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Is f continuous at all points of $\mathbb{R}^2$? Is f differentiable at all points of $\mathbb{R}^2$? Does f have directional derivatives at $(0, 0)$ in every direction? Justify all your answers. (3+3+4= 10 marks)

Solution: f is continuous at all points. Since $2xy \le x^2 + y^2$ we have $|\frac{xy}{\sqrt{x^2+y^2}}| \le |\frac{\sqrt{x^2+y^2}}{2}|$.
Hence the continuity for all points.
Note that the partial derivatives of f exists but does not coincide. $\partial_x f(x, y) = \frac{y^3}{(x^2+y^2)^{3/2}}$ and $\partial_y f(x, y) = \frac{x^3}{(x^2+y^2)^{3/2}}$. Moreover, the partial derivatives at the origin is zero. If the

derivative $D_f x \equiv 0$ exists at $(0,0)$, then $\lim \frac{|f(h_1,h_2)|}{|(h_1,h_2)|} = 0$. But $\frac{|f(h_1,h_2)|}{|(h_1,h_2)|} = \frac{1}{2}$ along the line $h_2 = h_1$ which is a contradiction.
Directional derivative in the direction $u = (u_1, u_2)$ at $(0,0)$ is given by

$$D_u f(0,0) = \lim_{h \to 0} \frac{f(0 + hu_1, 0 + hu_2) - f(0,0)}{h} = \frac{u_1 u_2}{\sqrt{u_1^2 + u_2^2}}$$

(5) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. Define $g : \mathbb{R}^n \to \mathbb{R}^{n+m}$ to be the map $g(x_1, \cdots, x_n) = (x_1, \cdots, x_n, f(x_1, \cdots, x_n))$. What is the derivative of g in terms of the derivative of f? Justify your answer. (6 marks)

Solution: Since f is a differentiable map, there exists a linear transformation $D_f : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$lim_{h \to 0} \frac{|f(x+h) - f(x) - D_f h|}{|h|} = 0.$$

Consider the linear map $D_g : \mathbb{R}^n \to \mathbb{R}^{n+m}$ defined by $D_g x = (x, D_f x)$. Then we have

$$\begin{aligned} \frac{|g(x+h) - g(x) - D_g h|}{|h|} &= \frac{|(0, \cdots, 0, f(x+h) - f(x) - D_f h)|}{|h|} \\ \implies \lim_{h \to 0} \frac{|g(x+h) - g(x) - D_g h|}{|h|} &= 0 \end{aligned}$$

(6) Let $E \subset \mathbb{R}^n$ be an open subset and let $f : E \to \mathbb{R}$ be a real valued function such that all the partial derivatives of f are bounded in E. Prove that f is continuous in E. (6 marks)

Given that the partial derivatives are bounded. Let $|\partial_i f| \leq M$ for all i. Now,

$$\begin{aligned} &|f(x_1 + h_1, \cdots .x_n + h_n) - f(x_1, \cdots, x_n)| \\ &= |f(x_1 + h_1, \cdots, x_n + h_n) - f(x_1, x_2 + h_2, \cdots, x_n + h_n) \\ &\quad + f(x_1, x_2 + h_2, \cdots, x_n + h_n) - f(x_1, x_2, x_3 + h_3, \cdots, x_n + h_n) + \cdots \\ &\quad + f(x_1, \cdots, x_{n-1}, x_n + h_n) - f(x_1, \cdots, x_{n-1}, x_n)| \end{aligned}$$

By mean value theorem, there exists a $t_i \in (x_i - \delta_i, x_i + \delta_i)$ for all i such that $(x_1 - \delta_1, x_1 + \delta_1) \times \cdots \times (x_n - \delta_n, x_n + \delta_n) \subset E$ and

$$\begin{aligned} &|f(x_1 + h_1, \cdots .x_n + h_n) - f(x_1, \cdots, x_n)| \\ &= |h_1 \partial_1 f(x_1, x_2 + h_2, \cdots, x_n + h_n) \\ &\quad + h_2 \partial_2 f(x_1, x_2, x_3 + h_3, \cdots, x_n + h_n) + \cdots \\ &\quad + h_n \partial f(x_1, \cdots, x_{n-1}, x_n)| \\ &\leq M(|h_1| + \cdots + |h_n|) \end{aligned}$$

Hence the continuity.